Dissipation in Turbulent Solutions of 2-D Euler

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Abstract

We establish local balance equations for smooth functions of the vorticity in the DiPerna-Majda weak solutions of 2D incompressible Euler, analogous to the balance proved by Duchon and Robert for kinetic energy in 3D. The anomalous term or defect distribution therein corresponds to the "enstrophy cascade" of 2D turbulence. It is used to define a rather natural notion of "dissipative Euler solution" in 2D. However, we show that the DiPerna-Majda solutions with vorticity in L^p for $p > 2$ are conservative and have zero defect. Instead, we must seek an alternative approach to dissipative solutions in 2D. If we assume an upper bound on the energy spectrum of 2D incompressible Navier-Stokes solutions by the Kraichnan-Batchelor k^{-3} spectrum, uniformly for high Reynolds number, then we show that the zero viscosity limits of the Navier-Stokes solutions exist, with vorticities in the zero-index Besov space $B_2^{0,\infty}$, and that these give a weak solution of the 2D incompressible Euler equations. We conjecture that for this class of weak solutions enstrophy dissipation may indeed occur, in a sense which is made precise.

1 Introduction

In 2-dimensional turbulence it is the *enstrophy* $\Omega(t) := \frac{1}{2}\|\omega(t)\|_2^2$ that is expected to cascade to small length-scales, not the energy as in three space dimensions [1]-[3]. In a view that goes back to Onsager [4], such turbulent cascades are conjectured to be described, in the limit of infinite Reynolds number, by singular (or weak) solutions of the incompressible Euler equations. More recently, Duchon and Robert [5] have shown how Onsager's idea of a dissipative Euler solution may be formalized in the three-dimensional case via a local energy balance relation. It is our purpose here to similarly formalize the notion of a 2-dimensional dissipative Euler solution, corresponding to the enstrophy cascade.

We consider weak solutions of the 2D Euler equations in the vorticity-velocity formulation:

$$\partial_t \omega + (\mathbf{u} \boldsymbol{\cdot} \boldsymbol{\nabla})\omega = 0, \tag{1.1}$$

with $\mathbf{u} = \mathbf{K} * \omega$ given by the Biot-Savart kernel $\mathbf{K}$. We show first that when the vorticity fields $\omega(\mathbf{x}, t)$ are suitable measurable functions and (1.1) is interpreted in the sense of distributions, then a local balance is satisfied

$$\partial_t h(\omega) + \boldsymbol{\nabla} \boldsymbol{\cdot} [\mathbf{u} h(\omega)] = -Z_h(\omega), \tag{1.2}$$

for nonnegative, convex functions $h(\omega)$. Of course, (1.1) formally just expresses the conservation of vorticity along fluid particle trajectories, so that (1.2) would naively be expected to hold with $Z_h(\omega) \equiv 0$. Dissipative weak solutions might be taken to be those for which this distribution is nonnegative: $Z_h(\omega) \geq 0$. The balance equation (1.2) makes more precise Polyakov's analogy of the enstrophy cascade with conservation law anomalies in quantum field-theory (such as the axial anomaly in QED) [6]. The distribution appearing as a sink term on the right side of (1.2) corresponds closely to such an anomaly. However, we show under rather general conditions, even weaker than those in our earlier work [7], that $Z_h(\omega) \equiv 0$. For example, we show that the anomaly vanishes for functions h of "power-p growth" at large arguments, whenever the initial vorticity satisfies an L^p bound in space. In particular, this means that *enstrophy is conserved*

by a 2D Euler solution whenever the enstrophy itself is finite. [1] This state of affairs presents a striking contrast with the situation in 3D where, as discussed by Duchon and Robert [5], energy dissipation is expected to be possible for incompressible Euler solutions with finite energy.

The above results necessitate an approach to the notion of dissipative Euler solution in the 2D case which is therefore rather different from that of Duchon-Robert for 3D. Nevertheless, 2D turbulence theory is still a useful guide to the correct formulation. Indeed, the above results are in perfect agreement with the expectations of the classical theories [1]-[3], which predict that the small-scale energy spectrum in the enstrophy cascade range of 2D turbulence shall be of the form $E(k) \sim Ck^{-3}$ (with at most a logarithmic correction). Hence, the classical theories of 2D turbulence predict an infinite total enstrophy but a finite spectral flux of enstrophy. In the following we shall formalize this notion for an appropriate class of weak Euler solutions in 2D. We define as dissipative those solutions which have a nonnegative flux of enstrophy (possibly zero or infinite) asymptotically to infinitely high wavenumbers. The relevant solutions must, however, have vorticity fields which exist only as distributions and not as ordinary (measurable) functions. We show that such solutions of 2D Euler equations with a Kraichnan-Batchelor k^{-3} energy spectrum are obtained as zero-viscosity limits of the Leray solutions of 2D Navier-Stokes, whenever upper bounds on the spectrum by the Kraichnan-Batchelor prediction hold uniformly in the viscosity. We then show that *the notion of enstrophy flux is well-defined for such distributional solutions of 2D Euler, although the enstrophy itself may diverge.* Our natural conjecture is that the flux is asymptotically nonnegative at small length-scales for all such "viscosity solutions" and, for suitable initial data, even strictly positive.

Our main results are stated as three Theorems in the following Section 2, where their content is further discussed in detail. The proofs of the Theorems are outlined in the final Section 3.

[1] This statement has, among other consequences, the implication that no power-law 2D energy spectrum $E(k) \sim Ck^{-n}$ with $n > 3$ may be compatible with existence of an enstrophy cascade. The conformal "solutions" studied by Polyakov [6] that have spectral exponent $n > 3$ therefore cannot exhibit an anomaly in the enstrophy conservation law, as he has proposed.

2 Statement of Results

Before stating precisely our theorems, it will help to motivate the statements (and the proofs) to give a brief, heuristic argument for the existence of the enstrophy cascade. In [7] we considered a "filtered" form of the 2-D Euler equations (see also [8]):

$$\partial_t \omega_\varepsilon + \boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}_\varepsilon \omega_\varepsilon + \boldsymbol{\sigma}_\varepsilon] = 0, \tag{2.1}$$

where $\omega_\varepsilon = \varphi_\varepsilon * \omega$ for a smooth mollifier φ, $\varphi_\varepsilon(\mathbf{x}) = \varepsilon^{-2}\varphi(\varepsilon^{-1}\mathbf{x})$, and $\boldsymbol{\sigma}_\varepsilon = (\mathbf{u}\omega)_\varepsilon - \mathbf{u}_\varepsilon\omega_\varepsilon$. The new term $\boldsymbol{\sigma}_\varepsilon$ represents a turbulent spatial transport of vorticity due to the eliminated small-scales. It is straightforward to show that the balance holds that

$$\partial_t h(\omega_\varepsilon) + \boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}_\varepsilon h(\omega_\varepsilon) + h'(\omega_\varepsilon)\boldsymbol{\sigma}_\varepsilon] = h''(\omega_\varepsilon)\boldsymbol{\nabla}\omega_\varepsilon\boldsymbol{\cdot}\boldsymbol{\sigma}_\varepsilon. \tag{2.2}$$

The term $Z_{h,\varepsilon}(\omega) := -h''(\omega_\varepsilon)\boldsymbol{\nabla}\omega_\varepsilon\boldsymbol{\cdot}\boldsymbol{\sigma}_\varepsilon$ represents a transfer of h-stuff from length-scales $> \varepsilon$ to smaller scales. Based upon the notion of "UV-locality of interactions", a natural approximation is to take $\boldsymbol{\sigma}_\varepsilon \approx (\text{const.})[(\mathbf{u}_\varepsilon\omega_\varepsilon)_\varepsilon - \mathbf{u}_\varepsilon\omega_\varepsilon]$ and then to Taylor expand to leading non-vanishing order to obtain

$$\boldsymbol{\sigma}_\varepsilon \approx C\varepsilon^2 \mathbf{D}_\varepsilon\boldsymbol{\cdot}\boldsymbol{\nabla}\omega_\varepsilon. \tag{2.3}$$

Here $\mathbf{D}_\varepsilon$ is the filtered velocity-gradient tensor $D_{ij} = \partial u_i/\partial x_j$; also, a spherically symmetric mollifier has been assumed. The first of our approximations is analogous to the "similarity model" employed by engineers in large-eddy simulation of three-dimensional turbulence and the second to its further simplification, the "nonlinear model" [9]. The matrix $\mathbf{D}_\varepsilon$ is traceless and has, in vortical regions of the flow, a pair of imaginary eigenvalues and, in strain-dominated regions, two real eigenvalues of equal magnitude S_ε but opposite signs. It stands to reason that, in the latter straining regions, the compression of vorticity level sets will tend to align the direction of the vorticity gradient $\boldsymbol{\nabla}\omega_\varepsilon$ with the eigendirection of $\mathbf{D}_\varepsilon$ corresponding to the negative eigenvalue. Indeed, such alignment has been observed in simulations to hold (for the unfiltered quantities) with a high probability [10]. Assuming it to hold exactly, we find that

$$\boldsymbol{\sigma}_\varepsilon \approx -C\varepsilon^2 S_\varepsilon \boldsymbol{\nabla}\omega_\varepsilon. \tag{2.4}$$

This is precisely an eddy-viscosity model, with effective viscosity $\nu_\varepsilon = C\varepsilon^2 S_\varepsilon$ at scale ε. It leads to an effective dissipation $Z_\varepsilon(\omega) \approx -\nu_\varepsilon |\boldsymbol{\nabla}\omega_\varepsilon|^2$. If the vorticity field is Hölder continuous with exponent s, $\omega \in C^s$, then $\boldsymbol{\nabla}\omega_\varepsilon \sim \varepsilon^{s-1}$ for small ε and $S_\varepsilon \sim S$ independent of ε. In that case, $Z_\varepsilon(\omega) \sim \varepsilon^{2s}$ for $\varepsilon \to 0$, so that we expect an asymptotic enstrophy cascade only when $s = 0$. This is precisely the "mean-field" scaling exponent in the Batchelor-Kraichnan theory [1]-[3].

We now state our main theorems:

Our first theorem establishes the local vorticity balance equations for the weak Euler solutions constructed by DiPerna and Majda for initial data $\omega_0 \in L^p, p > 1$ [11]. Although they considered solutions in the whole plane $\mathbb{R}^2$, we shall restrict attention for simplicity to solutions on the 2-D torus $\mathbb{T}^2$. DiPerna and Majda also established existence of weak solutions in the velocity-pressure formulation, but it is not hard to show that, for $p > 4/3$, the associated vorticity field in their solution also satisfies the weak vorticity-velocity equations (see below). In fact, the only property of the DiPerna-Majda solution that we will employ in our proof is that $\omega \in L^\infty([0,T], L^p(\mathbb{T}^2))$ and our theorem would apply to any other such solutions as well. To state our theorem, we must introduce an appropriate class of differentiable functions

$$\mathcal{H}_p := \big\{h \,|\; h \in C^1(\mathbb{R}),\;\; |h'(\omega)| \leq C(1 + |\omega|^{p-1}) \text{ for some } C > 0\big\} \tag{2.5}$$

which have at most L^p-growth. We then have the following:

Theorem 1 *If $\omega \in L^\infty([0,T], L^p(\mathbb{T}^2))$ and the associated $\mathbf{u} = \mathbf{K} * \omega$ for $p > 4/3$ are a weak solution of 2-D incompressible Euler in the vorticity-velocity formulation, then for $h \in \mathcal{H}_r \bigcap C^2$, with $r = \frac{3}{2}p - 1$ for $\frac{4}{3} < p < 2, r < p$ for $p = 2$, and $r = p$ for $p > 2$, the balance (1.2) holds*

$$\partial_t h(\omega) + \boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}h(\omega)] = -Z_h(\omega)$$

in the sense of distributions. The righthand side is given by the distributional limit

$$Z_h(\omega) = \lim_{\varepsilon \to 0} -h''(\omega_\varepsilon)\boldsymbol{\nabla}\omega_\varepsilon \boldsymbol{\cdot} \boldsymbol{\sigma}_\varepsilon \tag{2.6}$$

which exists for any choice of mollifier φ which is C^∞, nonnegative, and compactly supported, with unit integral, and it is independent of that choice. For the special case of the enstrophy

integral, $h(\omega)=\frac{1}{2}|\omega|^2$, when $p>2$, we write simply $Z(\omega)=Z_h(\omega)$. In that case, there is the alternative expression:

$$Z(\omega)=\lim_{\varepsilon\to 0}\frac{1}{4}\int d^2\ell\ \boldsymbol{\nabla}\varphi_\varepsilon(\ell)\boldsymbol{\cdot}\Delta_\ell\mathbf{u}|\Delta_\ell\omega|^2 \tag{2.7}$$

where $\Delta_\ell\omega(\mathbf{x},t)=\omega(\mathbf{x}+\ell,t)-\omega(\mathbf{x},t)$, likewise for $\Delta_\ell\mathbf{u}$, and φ is further restricted to be an even function of its argument.

Note that, formally, $Z_h(\omega)=h''(\omega)Z(\omega)$, so the fluxes of general convex functions are, in some sense, proportional to the enstrophy flux with a nonnegative factor. The last expression (2.7) for the enstrophy flux has a nice interpretation as a local, non-ensemble-averaged form of the "-2 law" for the direct cascade, in its form applicable without isotropy (see [7], Appendix B). Thus, the defect distribution in the vorticity balance equations has an exact connection with the enstrophy cascade in 2D turbulence theory.

However, we next show that this distribution is, in fact, zero for the DiPerna-Majda weak solutions, which therefore conserve the integral

$$I_h(t)=\int d^2\mathbf{x}\ h(\omega(\mathbf{x},t)) \tag{2.8}$$

for all h of suitable growth:

Theorem 2 *If $\omega\in L^\infty(0,T;L^p(\mathbb{T}^2))$ is a DiPerna-Majda weak Euler solution for $p\geq 2$, then*

$$\partial_t h(\omega)+\boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}h(\omega)]=0 \tag{2.9}$$

in distribution sense for all $h\in\mathcal{H}_r$ with $r=p$ when $p>2$ and for any $r<p$ when $p=2$.

In [7] it was proved that such a conservation statement holds for $\omega\in L^p(0,T;B_p^{s,\infty}(\mathbb{T}^2))$ for $s>0,p\geq 3$ where $B_p^{s,\infty}(\mathbb{T}^2)$ is the standard Besov space of functions in $L^p(\mathbb{T}^2)$ which are Hölder of index s in the L^p-mean sense [12]. That theorem was thus analogous to the Besov-space improvement of Onsager's original conservation result for 3D, which was proved by Constantin, E, and Titi [13]. We now see that the smoothness assumed in [7] was unnecessary and that simple L^p bounds alone are sufficient for conservation. Essentially the same result was already

obtained by P.-L. Lions in [14], Section 4.1, based upon his earlier work with R. J. DiPerna [15]. He showed there that the DiPerna-Majda solutions with $p > 2$ are "renormalized solutions" in the sense of DiPerna-Lions [15], which amounts to the requirement that (2.9) hold. In fact, global conservation

$$\int_{\mathbb{T}^2} d^2\mathbf{x}\ h(\omega(\mathbf{x},t)) = \int_{\mathbb{T}^2} d^2\mathbf{x}\ h(\omega_0(\mathbf{x})), \quad t > 0 \tag{2.10}$$

is shown in [14] to hold for all $h \in \mathcal{H}_p$ even when $p = 2$, just as in the proof of Theorem II.2 and equation (26) in DiPerna-Lions [15]. [2] In particular, taking $h(\omega) = \frac{1}{2}|\omega|^2$, a remarkable statement is true that *enstrophy dissipation is not possible for any 2D Euler solutions with finite enstrophy.* We conclude more generally that the DiPerna-Majda weak solutions are not relevant to the problem of constructing dissipative Euler solutions. In the language of turbulence theory, they do not support enstrophy cascades over infinitely-long ranges of wavenumber.

The conservation properties of the DiPerna-Majda solutions for $p > 2$ have an intuitive explanation. It has been noted recently that breakdown of uniqueness of Lagrangian particle trajectories in Hölder but non-Lipschitz flows can be a mechanism for the anomalous dissipation of the analogous integrals as (2.8) for passive scalars [17]-[19]. For the 3D problem, Shnirelman has found a weak solution which dissipates energy globally, by constructing a generalized flow with random Lagrangian trajectories [20]. In the case of the Yudovich solutions of 2D Euler with $\omega \in L^\infty(\mathbb{T}^2)$ [21], it has long been known that they are conservative precisely because the corresponding velocity field is log-Lipschitz and the Lagrangian flow maps X_t are unique, volume-preserving homeomorphisms. Therefore, the Yudovich solution is given simply by $\omega(\mathbf{x},t) = \omega_0(X_{-t}(\mathbf{x}))$ in terms of the inverse-Lagrangian map. All of the integrals $I_h(t)$ in (2.8) are then trivially time-invariant. DiPerna and Lions in their paper [15] show that there are likewise unique Lagrangian flow maps $X_t(x)$ with $X \in C(0,T;L^p(\mathbb{T}^2))$ whenever $\mathbf{u} \in L^1(0,T;W^{1,p}(\mathbb{T}^2))$ for $p \geq 1$ and that these maps preserve Lebesgue measure when $\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{u} = 0$. The "renormalized" solutions of the linear advection equation constructed by DiPerna-Lions

[2] The same remark was made in a recent preprint of E and Vanden-Eijnden [16].

are shown to have precisely the form $\omega(\mathbf{x},t) = \omega_0(X_{-t}(\mathbf{x}))$. While it is not true in general that the distributional solutions of 2D Euler in the sense considered here are renormalized solutions, Theorem 2, as we have observed above, shows that this is so for the DiPerna-Majda solutions when $p > 2$. Hence, the conservation properties of these solutions are again connected with the uniqueness of Lagrangian particle trajectories.

The above results are negative—in the sense that they imply a lack of enstrophy dissipation—but we wish to emphasize that they are fully consistent with the expectations of 2D turbulence theory. In fact, the Navier-Stokes solutions exhibiting an enstrophy cascade are expected to have the Batchelor-Kraichnan energy spectrum

$$E(k,t) \sim C\eta^{2/3}(t)k^{-3}, \tag{2.11}$$

where $\eta(t)$ is the enstrophy dissipation rate per volume [1, 3]. This spectrum should hold at high wavenumbers $k \gg k_0(t)$, the wavenumber of peak enstrophy, up to a wavenumber $k_d(t) = \nu^{-1/2}\eta^{1/6}(t)$, at which the dissipation by viscosity ν becomes relevant. Equivalently, the enstrophy spectrum predicted by Batchelor-Kraichnan theory is

$$\Omega(k,t) \sim C\eta^{2/3}(t)k^{-1}, \tag{2.12}$$

for $k_0(t) \ll k \ll k_d(t)$. In the limit as $\nu \to 0$ this spectrum extends all the way to $+\infty$ and its integral diverges, implying an infinite total enstrophy. As we have seen, this is rigorously required to have limiting Euler solutions which can dissipate enstrophy.

In fact, velocity fields $\mathbf{u}(t)$ with the Batchelor-Kraichnan spectrum (2.11) for all $k \gg k_0(t)$, when that spectrum is interpreted in a suitable sense, must consist of $\mathbf{u}(t) \in B_2^{1,\infty}(\mathbb{T}^2)$, with corresponding vorticity $\omega(t) \in B_2^{0,\infty}(\mathbb{T}^2)$, the Besov space of zero index. The definition of spectrum which is relevant is a "Littlewood-Paley spectrum" which was earlier used by P. Constantin in [22] to prove a rigorous upper bound. This spectrum is defined in terms of the Littlewood-Paley decomposition of the velocity $\mathbf{u}(t) = \sum_{N=0}^{\infty} \mathbf{u}_N(t)$ with $\mathbf{u}_N(t) = \psi_N * \mathbf{u}(t)$, for a smooth

partition of unity in wavenumber space

$$\widehat{\psi}_0(k) + \sum_{N=1}^{\infty} \widehat{\psi}_N(k) = 1 \tag{2.13}$$

where $\mathrm{supp}(\widehat{\psi}_N) \subset [2^{N-1}, 2^{N+1}]$ for $N \geq 1$ and $\mathrm{supp}(\widehat{\psi}_0) \subset [0, 2]$. The Littlewood-Paley spectrum is then defined by

$$E_{LP}(k,t) := k^{-1} \|\mathbf{u}_N(t)\|_{L^2}^2 \tag{2.14}$$

for $k \in [2^N, 2^{N+1})$. With this definition it is not hard to see that $\mathbf{u}(t) \in B_2^{1,\infty}(\mathbb{T}^2)$ precisely when the spectrum satisfies a bound of the form $E_{LP}(k,t) = O(k^{-3})$. In fact, the Littlewood-Paley criterion for $f \in B_p^{s,\infty}(\mathbb{T}^2)$ with $-\infty < s < \infty$ and $p > 0$ is just that

$$\|f\|_{B_p^{s,\infty}} := \sup_{N \geq 0} 2^{sN} \|f_N\|_{L^p} \tag{2.15}$$

be finite, and for $p \geq 1$ this is a norm making $B_p^{s,\infty}(\mathbb{T}^2)$ into a Banach space. See [12], Sections 2.3.1-3. Note we assume only big-O bounds on the spectrum and not a power-law scaling. In fact, not long after his first paper on 2D turbulence, Kraichnan argued that there should be a logarithmic correction, $E(k,t) \sim C\eta^{2/3}(t)k^{-3}[\ln(k/k_0)]^{-\frac{1}{3}}$, where k_0 is the lower end of the enstrophy cascade range [23]. In any case, it is still generally believed that the true energy spectrum must be bounded above by the form (2.11) at high Reynolds number. In that case, we see that the vorticity field $\omega(t)$ has the Besov index $s = 0$ but not necessarily any larger index. As we have already remarked, $\omega(t) \in B_p^{s,\infty}(\mathbb{T}^2)$ with $p > 2$ and $s > 0$ could not be consistent with a non-vanishing enstrophy dissipation.

It is still an open question whether solutions of 2D Euler equations exist with velocities and vorticities in such Besov spaces and, if so, whether they dissipate enstrophy in a suitable sense. We shall prove here a few simple results in this direction and, in particular, advance our main conjecture.

We show first that an upper bound on the energy spectrum of the solutions of the 2D Navier-Stokes solutions $\mathbf{u}^\nu(t)$ by the Batchelor-Kraichnan spectrum, when that bound is uniform in the viscosity, implies the existence of 2D Euler solutions $\mathbf{u}(t)$ in the appropriate Besov

spaces. In [22] Constantin proved that the long-time average energy spectrum, $\overline{E}^{\nu}_{LP}(k) = \limsup_{T\to\infty} \frac{1}{T}\int_0^T dt\ E^{\nu}_{LP}(k,t)$ of the 2D Navier-Stokes solutions satisfies a bound of the form

$$\overline{E}^{\nu}_{LP}(k) \leq C\gamma^2 k^{-3}\left(\frac{k_d}{k}\right)^6, \tag{2.16}$$

where $\gamma = \|\boldsymbol{\nabla}\mathbf{u}^{\nu}\|_{L^\infty}$. This upper bound is much larger than the Kraichnan-Batchelor spectrum (2.11) over most of the range $k_0 < k < k_d$, but becomes comparable at the upper limit. We are going to assume here that something stronger is true of the 2D Navier-Stokes solutions, namely, for some $T > 0$:

$$\sup_{\nu>0}\frac{1}{T}\int_0^T dt\ \sup_{k>k_0} k^3 E^{\nu}_{LP}(k,t) < \infty \tag{2.17}$$

Note that $C^{\nu}(t) = \sup_{k>k_0} k^3 E^{\nu}_{LP}(k,t)/\eta^{2/3}$ is an instantaneous (worst) value of the Kraichnan-Batchelor constant and (2.17) is a bound on its time-average. The most important aspect of this estimate—in contrast to what is so far proved, equation (2.16)—is its uniformity for small viscosity $\nu > 0$. Our main hypothesis (2.17) is equivalent to

$$\sup_{\nu>0}\|\mathbf{u}^{\nu}\|_{L^2(0,T;B^{1,\infty}_2(\mathbb{T}^2))} < \infty \tag{2.18}$$

Using estimates for singular integral operators, this may also be expressed equivalently in terms of the vorticity $\omega^{\nu} = \boldsymbol{\nabla}\times\mathbf{u}^{\nu}$, as

$$\sup_{\nu>0}\|\omega^{\nu}\|_{L^2(0,T;B^{0,\infty}_2(\mathbb{T}^2))} < \infty \tag{2.19}$$

This latter estimate could be stated in terms of the Littlewood-Paley enstrophy spectrum $\Omega^{\nu}_{LP}(k,t) := k^{-1}\|\omega^{\nu}_N(t)\|^2_{L^2}$ for $k \in [2^N, 2^{N+1})$, that

$$\sup_{\nu>0}\frac{1}{T}\int_0^T dt\ \sup_{k>k_0} k\Omega^{\nu}_{LP}(k,t) < \infty, \tag{2.20}$$

which is entirely equivalent to the initial hypothesis (2.17).

We now state our third main result:

Theorem 3 *Let $\mathbf{u}^{\nu}$ be the solution of the 2D incompressible Navier-Stokes equation for initial data $\mathbf{u}_0 \in B^{1,\infty}_2(\mathbb{T}^2)$ and viscosity $\nu > 0$. Let $\omega^{\nu} = \boldsymbol{\nabla}\times\mathbf{u}^{\nu}$. Assume that (2.18) holds for these*

solutions. Then, there exists a $\mathbf{u} \in L^2(0,T;B_2^{1,\infty}(\mathbb{T}^2)) \cap \mathrm{Lip}(0,T;H^{-L}(\mathbb{T}^2))$ *which is a weak solution of the 2D incompressible Euler equations in the velocity-pressure formulation, and for which, with* $\omega = \boldsymbol{\nabla}\times\mathbf{u}$,

$$\omega^\nu \rightharpoonup \omega \ \ weak - * \ \text{ in } \ L^2(0,T;B_2^{0,\infty}(\mathbb{T}^2)), \tag{2.21}$$

$$\omega^\nu \to \omega \ \ strong \ \ in \ \ L^2(0,T;W^{-s,q}(\mathbb{T}^2)), \tag{2.22}$$

for some $q > 2$ *and* $s > 1 - \frac{2}{q}$*, in the limit as* $\nu \to 0$*. Furthermore,* ω *and* $\mathbf{u} = \mathbf{K} * \omega$ *are a weak solution of the 2D incompressible Euler equations in the vorticity-velocity formulation, in the sense that*

$$\langle(\partial_t + \mathbf{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\psi, \omega\rangle = 0 \tag{2.23}$$

for all $\psi \in C_0^\infty([0,T]\times\mathbb{T}^2)$ *and the expression* $\langle(\partial_t + \mathbf{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\psi, \omega\rangle$ *is defined as the evaluation of a continuous linear functional on the element* ω *of the Banach space* $L^2(0,T;B_2^{0,\infty}(\mathbb{T}^2))$.

This theorem essentially just states that the estimate (2.18) provides enough compactness to take limits along subsequences. Obviously, the hard problem is to prove that a bound such as (2.18), as expected from 2D turbulence theory, really does hold. The theorem could be stated in a somewhat more general form, with the results on weak solutions in the velocity-pressure formulation remaining true for any $p > 1$ replacing $p = 2$, if a corresponding replacement is made in the estimate (2.18). Likewise, the results on weak solutions in the vorticity-velocity formulation will remain true for any $p > 4/3$ replacing $p = 2$.

Our interest in this class of solutions is that they seem compatible with a finite rate of enstrophy dissipation in the inviscid limit. However, the very notion of "dissipative solution" must be reformulated. Local functions of the vorticity, of the form $h(\omega(\mathbf{x},t))$, do not need even to exist, since now the vorticity ω is only a distribution and not necessarily a measurable function. Thus, a balance equation such as (1.2) that we proved in Theorem 1 for DiPerna-Majda solutions is not even well-defined for the class of solutions considered here. However,

the balance equations for the mollified vorticity in (2.2), namely,

$$\partial_t h(\omega_\varepsilon) + \boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}_\varepsilon h(\omega_\varepsilon) + h'(\omega_\varepsilon)\boldsymbol{\sigma}_\varepsilon] = -Z_{h,\varepsilon}(\omega) \tag{2.24}$$

with $Z_{h,\varepsilon}(\omega) = -h''(\omega_\varepsilon)\boldsymbol{\nabla}\omega_\varepsilon\boldsymbol{\cdot}\boldsymbol{\sigma}_\varepsilon$, are still perfectly well-defined. The term $Z_{h,\varepsilon}(\omega)$ which appears as sink on the righthand side of (2.24) represents a flux of h to length-scales $< \varepsilon$ and it is expected to be asymptotically non-negative for small ε. In fact, more should be true. A corresponding balance equation holds for the solutions of the 2D Navier-Stokes solutions ω^ν, in the form

$$\partial_t h(\omega^\nu) + \boldsymbol{\nabla}\boldsymbol{\cdot}[h(\omega^\nu)\mathbf{u}^\nu - \nu\boldsymbol{\nabla} h(\omega^\nu)] = -\nu h''(\omega^\nu)|\boldsymbol{\nabla}\omega^\nu|^2, \tag{2.25}$$

for any $h \in C^2$. Then we expect the following

Conjecture 1 *Let ω^ν be a sequence of solutions of the 2D Navier-Stokes equation obeying (2.18) and let ω be the limiting 2D Euler solution, as provided by Theorem 3. Then, for this Euler solution*

$$Z_h(\omega) = \lim_{\varepsilon\to 0} Z_{h,\varepsilon}(\omega) \tag{2.26}$$

exists in the sense of distributions for any $h \in \mathcal{H}_2$. Furthermore, the same distribution is obtained by the limit of the viscous dissipation of the Navier-Stokes solutions:

$$Z_h(\omega) = \lim_{\nu\to 0} \nu h''(\omega^\nu)|\boldsymbol{\nabla}\omega^\nu|^2 \tag{2.27}$$

for any $h \in \mathcal{H}_2 \bigcap C^2$.In particular, for any such convex h, the distribution $Z_h(\omega)$ is a nonnegative measure. Finally, there should exist a suitable such 2D Euler solution ω for which

$$Z_h(\omega) > 0 \tag{2.28}$$

with a strict inequality, for a convex $h \in \mathcal{H}_2$.

The first limit statement in the conjecture may be put another way, which is perhaps more illuminating. Although the integral $I_h(t)$ may itself be infinite for the Euler solutions in Theorem 3, it still makes sense to talk about a finite *dissipation rate* for it, defined as $D_h(t) :=$

$\liminf_{\varepsilon\to 0} -\frac{dI_h^\varepsilon}{dt}(t)$, where $I_h^\varepsilon(t)$ is the value of the integral for ω_ε. The conjecture then states $D_h(t) = \int_{\mathbb{T}^2} d^2\mathbf{x}\ Z_h(\omega)(\mathbf{x},t) > 0$. Note that for DiPerna-Majda solutions the first limit (2.26) has been demonstrated in Theorem 1 and it is easy to show for these solutions that the second limit (2.27) also holds, using the same kind of argument as in Proposition 4 of Duchon-Robert [5]. Of course, for DiPerna-Majda solutions with $\omega \in L^p$ and $p > 2$ the distribution $Z_h(\omega) \equiv 0$ and thus the third statement is false.

We believe that it is necessary to understand solutions of the type considered in Theorem 3 in order to develop a rigorous mathematical theory of invariant measures for forced steady-states of 2D Navier-Stokes in the zero-viscosity limit. As proved in [7], Section 3.3.4, the mean enstrophy flux $\langle Z_\varepsilon\rangle$ is a positive constant η, independent of ε, for length-scales $\varepsilon \ll \ell_f$, the forcing scale, and $\varepsilon \gg \nu^{1/4}E^{1/4}/\eta^{1/4}$, under the single assumption that the total mean energy E remains finite in the limit as $\nu \to 0$. (This requires adding an additional dissipation at low-wavenumbers to dispose of the "condensate" from the inverse energy cascade: see [7]). Thus, in the limit as $\nu \to 0$, we expect that the realizations of the ensemble shall be solutions of the (forced) 2D Euler equations with $Z(\omega) > 0$. If the statistical energy spectrum has the Batchelor-Kraichnan form in this limit, then individual realizations of the vorticity satisfy $\omega \in B_2^{0-,\infty}(\mathbb{T}^2)$ a.s. It is not hard to prove this fact, using the methods of [24] (the wavelet characterization of Besov spaces and the Borel-Cantelli argument of Theorem 4).

To see dissipation in the sense of our Conjecture 1 for the problem of free decay of 2D turbulence starting from random initial conditions, one must begin with initial data which is sufficiently rough. It is well-known that if one starts with $\omega_0 \in B_p^{s,\infty}(\mathbb{T}^2)$ for $s > 0$ at time $t = 0$, then the exponent may (and generally will) deteriorate exponentially in time: for example, $s(t) = e^{-C\|\omega_0\|_\infty t}s$ if $\omega_0 \in L^\infty(\mathbb{T}^2) \bigcap B_p^{s,\infty}(\mathbb{T}^2)$, but the exponent remains positive [25]. Thus, there will be no dissipation at any finite time. On the other hand, the deterioration is consistent with the expectation from 2D turbulence theory that there will be an exponentially growing range of scales ε with $\langle Z_\varepsilon(t)\rangle \approx \eta(t)$, independent of ε [26, 27]. To see dissipation at finite (or zero) time, one must begin with initial data no more regular than $\omega_0 \in B_p^{0,\infty}(\mathbb{T}^2)$ a.s.

for $p \geq 2$. Such initial data could be prepared, for example, by taking an invariant measure for the driven problem and then turning off the force. The results of DiPerna and Lions [15] do not rule out dissipation in this instance, because they require $\mathbf{u} \in L^1(0,T;W^{1,p}(\mathbb{T}^2))$ for some $p \geq 1$, whereas $W^{1,p}(\mathbb{T}^2) = B_p^{1,p}(\mathbb{T}^2) \subsetneq B_p^{1,\infty}(\mathbb{T}^2)$. If $\mathbf{u}(t) \in B_p^{1,\infty}(\mathbb{T}^2)$ only, then examples like that in section IV.2 of [15] show that uniqueness of the Lagrangian trajectories breaks down and dissipation (in the sense of non-vanishing enstrophy flux) is possible.

It is natural to expect that 2D Euler solutions which are dissipative in the proposed sense, i.e. $Z_h(\omega) \geq 0$ for convex h, must be unique. Our Conjecture 1 states that "viscosity solutions" of 2D Euler equations are dissipative, so that these must then also be unique. Duchon and Robert [5] have advanced the same idea for the 3D case. There is perhaps even more reason to believe so in 2D, because there is then an infinity of convex "entropies" h. For the problem of scalar conservation laws, such entropies play a crucial role in establishing uniqueness (e.g. see [28]). However, unlike the scalar case, it is not necessarily true even for smooth classical solutions of 2D Euler that the dynamics is L^1-contractive. In fact, for two such solutions ω_1, ω_2, $\frac{d}{dt}\|\omega_1(t) - \omega_2(t)\|_1 = -2\int_{\omega_1=\omega_2} \mathbf{n}_{12}\boldsymbol{\cdot}(\mathbf{u}_1 - \mathbf{u}_2)\omega \, ds$ where $\mathbf{n}_{12}$ is the unit vector normal to the curve $\omega_1 = \omega_2$ from the region $\omega_1 > \omega_2$ to $\omega_2 > \omega_1$, and $\omega = \omega_1 = \omega_2$. It is precisely the nonlocal relation between $\mathbf{u}$ and ω which allows $\mathbf{u}_1 \neq \mathbf{u}_2$ where $\omega_1 = \omega_2$. So far, uniqueness of weak Euler solutions in 2D is established only for the solutions with $\omega \in L^\infty([0,T] \times \mathbb{T}^2)$ constructed by Yudovich [21] and for DiPerna-Majda solutions in $L^p, p \in (1,\infty)$ if also $\omega \in BMO$ [29]. It is not known in general whether the DiPerna-Majda solutions are unique, although for $p > 2$ they are "dissipative Euler solutions", in the sense that

$$\partial_t h(\omega) + \boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}h(\omega)] \leq 0 \tag{2.29}$$

for all convex $h \in \mathcal{H}_p$. In fact, as noted above, DiPerna-Majda solutions for $p > 2$ are "renormalized solutions" as considered by DiPerna-Lions and satisfy (2.29) in the degenerate sense with equality. Yet their uniqueness is an open question.

3 Proofs

2.1. Proof of Theorem 1

We comment first on the validity of the weak vorticity-velocity equation for the DiPerna-Majda solutions. The condition $p > 4/3$ arises from the requirement that the nonlinear advection term $\mathbf{u}\omega \in L^1(\mathbb{T}^2)$. Since $\mathbf{u} \in W^{1,p}(\mathbb{T}^2) \subset L^{p'}(\mathbb{T}^2)$ for $\frac{1}{p'} = \frac{1}{p} - \frac{1}{2}$ by Sobolev imbedding, one finds that $p' > q$, with q defined by $\frac{1}{q} = 1 - \frac{1}{p}$, when $p > 4/3$. Then $\mathbf{u}\omega \in L^1(\mathbb{T}^2)$ follows by Hölder inequality. The weak velocity-pressure form of the Euler equation is that

$$\int d^2\mathbf{x} \int dt \; [\partial_t \boldsymbol{\phi}\boldsymbol{\cdot}\mathbf{u} + \boldsymbol{\nabla} \otimes \boldsymbol{\phi} : \mathbf{u} \otimes \mathbf{u}] = 0 \tag{3.1}$$

for any smooth, divergence-free test function $\boldsymbol{\phi}(\mathbf{x}, t)$. In particular, $\boldsymbol{\phi} = \boldsymbol{\nabla}^\perp \psi$ satisfies these conditions for any smooth ψ, where $\boldsymbol{\nabla}^\perp$ is the skew-gradient, $\partial_i^\perp = \varepsilon_{ij}\partial_j$ with ε_{ij} the Levi-Civita tensor in 2D. (In fact, by Hodge theory, any divergence-free vector field $\boldsymbol{\phi}$ in 2D can be written in this way.) Substituting $\boldsymbol{\phi} = \boldsymbol{\nabla}^\perp \psi$ into (3.1) it is easy, using the L^1 property of $\mathbf{u}\omega$ and $\omega = -\boldsymbol{\nabla}^\perp\boldsymbol{\cdot}\mathbf{u}$, to derive the vorticity-velocity equation by an approximation argument.

The main condition of Theorem 1 on the index r can be similarly understood from the following lemma:

Lemma 1 *If $\omega \in L^p(\mathbb{T}^2)$ for $p > 4/3$ and $\mathbf{u} = \mathbf{K} * \omega$, then for any $h \in \mathcal{H}_r$ it holds that $\mathbf{u}h(\omega) \in L^1(\mathbb{T}^2)$ when $r = \frac{3}{2}p - 1$ for $\frac{4}{3} < p < 2$, $r < p$ for $p = 2$, and $r = p$ for $p > 2$.*

Remark: For convenience in the proof below, and in all later proofs, we employ an equivalent definition of the class of functions

$$\mathcal{H}_p := \big\{ h | \;\; h \in C^1(\mathbb{R}), \;\; |h'(\omega)| \le C|\omega|^{p-1} \text{ for } |\omega| \ge R \text{ for some } C, R > 0 \big\} \tag{3.2}$$

We will make the argument then assuming that $R = 0$ so that the bound in (3.2) above holds globally. In fact, when $R > 0$ it is easy to bound the contributions from the small-ω regions of integration over space and time by terms proportional to $\|h'\|_{L^\infty[-R,R]}, \|h''\|_{L^\infty[-R,R]}$, assuming that the latter are finite. So we lose no generality and simplify the arguments by taking $R = 0$.

Proof of Lemma: We first note the definition $\mathbf{K} := \boldsymbol{\nabla}^{\perp} G$ where G is the Greens function of $-\triangle$ on $\mathbb{T}^2$. Then $\mathbf{u} \in W^{1,p}(\mathbb{T}^2)$ because $\|\mathbf{u}\|_p \leq \|\mathbf{K}\|_1 \|\omega\|_p$ by Young's inequality and $\|\boldsymbol{\nabla}\mathbf{u}\|_p \leq C\|\omega\|_p$ by the Calderón-Zygmund inequality. Hence, by the same Sobolev imbedding as before, $\mathbf{u} \in L^{p'}(\mathbb{T}^2)$ for $\frac{1}{p'} = \frac{1}{p} - \frac{1}{2}$ when $\frac{4}{3} < p < 2$ and for any *finite* $p' \geq 1$ when $p = 2$, and for $p' = \infty$ when $p > 2$. Then, by definition of $\mathcal{H}_r$,

$$\|\mathbf{u}h(\omega)\|_1 \leq (\text{const.})\|\mathbf{u}|\omega|^r\|_1 \leq (\text{const.})\|\mathbf{u}\|_{p'}\|\omega\|_{rq'}^r \tag{3.3}$$

with $\frac{1}{q'} = 1 - \frac{1}{p'}$. When $\frac{4}{3} < p < 2$, then $\frac{1}{q'} = \frac{3}{2} - \frac{1}{p}$ and $rq' = p$ for $r = \frac{3}{2}p - 1$. On the other hand, when $p > 2$, then $q' = 1$, and $rq' = p$ for $r = p$. Lastly, in the critical case $p = 2$, the only requirement is that $q' > 1$. Then $rq' \leq p$ can be satisfied for any $r < p$ by an appropriate choice of $q' > 1$. Thus, for the given definitions of r,

$$\|\mathbf{u}h(\omega)\|_1 \leq (\text{const.})\|\omega\|_p^{r+1} \tag{3.4}$$

because $\|\mathbf{u}\|_{p'} \leq C\|\mathbf{u}\|_{W^{1,p}} \leq C'\|\omega\|_p$. $\Box$

Proof of Theorem 1: We consider the filtered balance equation (2.2):

$$\partial_t h(\omega_\varepsilon) + \boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}_\varepsilon h(\omega_\varepsilon) + h'(\omega_\varepsilon)\boldsymbol{\sigma}_\varepsilon] = h''(\omega_\varepsilon)\boldsymbol{\nabla}\omega_\varepsilon\boldsymbol{\cdot}\boldsymbol{\sigma}_\varepsilon.$$

and, just as in [5], we show that every term on the lefthand side has a limit in the sense of distributions for $\varepsilon \to 0$. We show first that $h(\omega_\varepsilon) \to h(\omega)$. In fact, by the mean-value theorem, $h(\omega_\varepsilon) - h(\omega) = h'(\bar{\omega}_\varepsilon)(\omega_\varepsilon - \omega)$ for $\bar{\omega}_\varepsilon(\mathbf{x}, t) = \lambda(\mathbf{x}, t)\omega(\mathbf{x}, t) + (1 - \lambda(\mathbf{x}, t))\omega_\varepsilon(\mathbf{x}, t)$ with some $0 \leq \lambda(\mathbf{x}, t) \leq 1$. Then, in the notations of Lemma 1, we have

$$|h(\omega_\varepsilon(\mathbf{x}, t)) - h(\omega(\mathbf{x}, t))| \leq (\text{const.})|\bar{\omega}_\varepsilon(\mathbf{x}, t)|^{r-1}|\omega_\varepsilon(\mathbf{x}, t) - \omega(\mathbf{x}, t)|$$

and thus by Hölder inequality

$$\begin{aligned} \|h(\omega_\varepsilon(t)) - h(\omega(t))\|_{q'} &\leq (\text{const.})\|\bar{\omega}_\varepsilon(t)\|_{rq'}^{r-1}\|\omega_\varepsilon(t) - \omega(t)\|_{rq'} \\ &\leq (\text{const.})\|\omega(t)\|_p^{r-1}\|\omega_\varepsilon(t) - \omega(t)\|_p. \end{aligned} \tag{3.5}$$

By the properties of the mollifier, $\lim_{\varepsilon\to 0}\|\omega_\varepsilon(t)-\omega(t)\|_p = 0$ for a.e. $t\in[0,T]$, and thus $\lim_{\varepsilon\to 0}\|h(\omega_\varepsilon(t))-h(\omega(t))\|_{q'}=0$. To complete the argument, we use the uniform bound

$$\|h(\omega_\varepsilon(t))-h(\omega(t))\|_{q'} \le (\mathrm{const.})\|\omega\|^r_{L^\infty(0,T;L^p(\mathbb{T}^2))} \tag{3.6}$$

to conclude by dominated convergence that $\lim_{\varepsilon\to 0}\|h(\omega_\varepsilon)-h(\omega)\|_{L^{q'}([0,T]\times\mathbb{T}^2)}=0$, which implies convergence $h(\omega_\varepsilon)\to h(\omega)$ in sense of distributions.

We show next for the middle term that $\mathbf{u}_\varepsilon h(\omega_\varepsilon)\to \mathbf{u}h(\omega)$. In fact, with notations again as in Lemma 1,

$$\begin{aligned}\|\mathbf{u}_\varepsilon(t)h(\omega_\varepsilon(t))-\mathbf{u}(t)h(\omega(t))\|_1 &\le \|\mathbf{u}_\varepsilon(t)-\mathbf{u}(t)\|_{p'}\|h(\omega_\varepsilon(t))\|_{q'}+\|\mathbf{u}(t)\|_{p'}\|h(\omega_\varepsilon(t))-h(\omega(t))\|_{q'}\\ &\le (\mathrm{const.})\|\mathbf{u}_\varepsilon(t)-\mathbf{u}(t)\|_{p'}\|\omega(t)\|_p^r+\|\mathbf{u}(t)\|_{p'}\|h(\omega_\varepsilon(t))-h(\omega(t))\|_{q'}. \end{aligned}\tag{3.7}$$

Thus, we see that $\lim_{\varepsilon\to 0}\|\mathbf{u}_\varepsilon h(\omega_\varepsilon(t))-\mathbf{u}h(\omega(t))\|_1=0$ for a.e. $t\in[0,T]$. In this case we have the uniform bound

$$\|\mathbf{u}_\varepsilon h(\omega_\varepsilon(t))-\mathbf{u}h(\omega(t))\|_1 \le (\mathrm{const.})\|\omega\|^{r+1}_{L^\infty(0,T;L^p(\mathbb{T}^2))} \tag{3.8}$$

so that we can use Lebesgue's theorem again to infer $\lim_{\varepsilon\to 0}\|\mathbf{u}_\varepsilon h(\omega_\varepsilon)-\mathbf{u}h(\omega)\|_{L^1([0,T]\times\mathbb{T}^2)}=0$, which gives the result.

Finally, for the third term we show that $h'(\omega_\varepsilon)\boldsymbol{\sigma}_\varepsilon\to\mathbf{0}$ as a distribution. We use the definition $\boldsymbol{\sigma}_\varepsilon=(\mathbf{u}\omega)_\varepsilon-\mathbf{u}_\varepsilon\omega_\varepsilon$ and the Hölder inequality

$$\|h'(\omega_\varepsilon(t))\boldsymbol{\sigma}_\varepsilon(t)\|_1 \le \|h'(\omega_\varepsilon(t))\|_{p/(r-1)}\|(\mathbf{u}(t)\omega(t))_\varepsilon-\mathbf{u}_\varepsilon(t)\omega_\varepsilon(t)\|_{p/(p-r+1)} \tag{3.9}$$

along with $\|h'(\omega_\varepsilon(t))\|_{p/(r-1)}\le(\mathrm{const.})\|\omega_\varepsilon(t)\|_p^{r-1}$ and the triangle inequality

$$\begin{aligned}\|(\mathbf{u}(t)\omega(t))_\varepsilon-\mathbf{u}_\varepsilon(t)\omega_\varepsilon(t)\|_{p/(p-r+1)} &\le \|(\mathbf{u}(t)\omega(t))_\varepsilon-\mathbf{u}(t)\omega(t)\|_{p/(p-r+1)}\\ &+\|\mathbf{u}(t)\|_{p'}\|\omega(t)-\omega_\varepsilon(t)\|_p+\|\mathbf{u}(t)-\mathbf{u}_\varepsilon(t)\|_{p'}\|\omega_\varepsilon(t)\|_p \end{aligned}\tag{3.10}$$

to infer that $\lim_{\varepsilon\to 0}\|(\mathbf{u}(t)\omega(t))_\varepsilon-\mathbf{u}_\varepsilon(t)\omega_\varepsilon(t)\|_{p/(p-r+1)}=0$ for a.e. $t\in[0,T]$. Note that we have used $\|\mathbf{u}(t)\omega(t)\|_{p/(p-r+1)}\le\|\mathbf{u}(t)\|_{p/(p-r)}\|\omega(t)\|_p$ and $(p-r)/p<p'$. Again a uniform

bound on $\|h'(\omega_\varepsilon(t))\boldsymbol{\sigma}_\varepsilon(t)\|_1$ like that in (3.8) completes the argument. Gathering these results, we see that the entire lefthand side of (2.2) approaches $\partial_t h(\omega) + \boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}h(\omega)]$ in the sense of distributions as $\varepsilon \to 0$. Obviously this limit is independent of the mollifier φ and the righthand side $-Z_{h,\varepsilon}(\omega) = h''(\omega_\varepsilon)\boldsymbol{\nabla}\omega_\varepsilon\boldsymbol{\cdot}\boldsymbol{\sigma}_\varepsilon$ has the same limit. This gives the first half of Theorem 1.

The second half of the theorem for the particular choice $h(\omega) = \frac{1}{2}|\omega|^2$ follows by the same argument as in [5]. In this proof, the balance (2.2) is replaced by

$$\partial_t(\frac{1}{2}\omega\omega_\varepsilon) + \boldsymbol{\nabla}\boldsymbol{\cdot}[(\frac{1}{2}\omega\omega_\varepsilon)\mathbf{u}] = -\widetilde{Z}_\varepsilon(\omega) \tag{3.11}$$

where an easy calculation gives

$$\widetilde{Z}_\varepsilon(\omega) := \frac{1}{2}\omega\boldsymbol{\nabla}\boldsymbol{\cdot}[(\omega\mathbf{u})_\varepsilon] - \frac{1}{2}\omega(\mathbf{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\omega_\varepsilon. \tag{3.12}$$

An argument exactly like the previous one shows that, when $p > 2$, the distributional limit $\lim_{\varepsilon\to 0}\widetilde{Z}_\varepsilon(\omega)$ exists and equals $-\partial_t(\frac{1}{2}|\omega|^2) - \boldsymbol{\nabla}\boldsymbol{\cdot}[(\frac{1}{2}|\omega|^2)\mathbf{u}] = Z(\omega)$. In addition, a simple calculation using the incompressibility of the velocity field shows that the expression appearing in (2.7) in Theorem 1 can be written

$$\int d^2\ell\ \boldsymbol{\nabla}\varphi_\varepsilon(\ell)\boldsymbol{\cdot}\Delta_\ell\mathbf{u}|\Delta_\ell\omega|^2 = \boldsymbol{\nabla}\boldsymbol{\cdot}[\mathbf{u}(\omega^2)_\varepsilon - (\mathbf{u}\omega^2)_\varepsilon] + 4\widetilde{Z}_\varepsilon(\omega). \tag{3.13}$$

As before, it is easy to show for $p > 2$ that $\mathbf{u}(\omega^2)_\varepsilon - (\mathbf{u}\omega^2)_\varepsilon \to \mathbf{0}$ as a distribution when $\varepsilon \to 0$. Hence, it follows that the limits of $\frac{1}{4}\int d^2\ell\ \boldsymbol{\nabla}\varphi_\varepsilon(\ell)\boldsymbol{\cdot}\Delta_\ell\mathbf{u}|\Delta_\ell\omega|^2$ and $\widetilde{Z}_\varepsilon(\omega)$ are also the same. That proves the second half of Theorem 1. □

2.2. Proof of Theorem 2

A result on global conservation corresponding to the local result in Theorem 2 was already proved in [7] but with an additional smoothness assumption that $\omega \in L^p(0,T;B_p^{s,\infty}(\mathbb{T}^2))$. Here we show that conservation holds without any such a smoothness assumption. Let $\tau_\varepsilon(f,g) := (fg)_\varepsilon - f_\varepsilon g_\varepsilon$ where $f_\varepsilon = \varphi_\varepsilon * f$. Then, we make use of the following key estimate:

Lemma 2 *Let $\omega \in L^p(\mathbb{T}^2)$ and $\mathbf{u} \in W^{1,p}(\mathbb{T}^2)$ for $p \geq 2$, and let $\boldsymbol{\nabla}\boldsymbol{\cdot}\mathbf{u} = 0$. Then*

$$\|\boldsymbol{\nabla}\boldsymbol{\cdot}\tau_\varepsilon(\mathbf{u},\omega)\|_{L^{p/2}} \leq C\|\mathbf{u}\|_{W^{1,p}}\|\omega\|_{L^p} \tag{3.14}$$

with a constant C independent of ε.

Proof: Note that

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\tau_\varepsilon(\mathbf{u},\omega) = \boldsymbol{\nabla}\boldsymbol{\cdot}[(\mathbf{u}\omega)_\varepsilon - \mathbf{u}\omega_\varepsilon] + (\mathbf{u}-\mathbf{u}_\varepsilon)\boldsymbol{\cdot}\boldsymbol{\nabla}\omega_\varepsilon. \tag{3.15}$$

The first term is handled in exactly the same manner as in Lemma II.1 of [15]. However, it is easy to see that

$$\|\mathbf{u}-\mathbf{u}_\varepsilon\|_{L^p} \le \varepsilon\|\boldsymbol{\nabla}\mathbf{u}\|_{L^p} \le \varepsilon\|\mathbf{u}\|_{W^{1,p}} \tag{3.16}$$

and

$$\|\boldsymbol{\nabla}\omega_\varepsilon\|_{L^p} \le \varepsilon^{-1}\|\boldsymbol{\nabla}\varphi\|_{L^1}\|\omega\|_{L^p}. \tag{3.17}$$

These control the second term. □

Corollary 1 *Under the same hypotheses, let $r_\varepsilon := -\boldsymbol{\nabla}\boldsymbol{\cdot}\tau_\varepsilon(\mathbf{u},\omega)$. Then $\lim_{\varepsilon\to 0} r_\varepsilon = 0$ strong in $L^{p/2}(\mathbb{T}^2)$ for $p \ge 2$.*

Proof: Since $\lim_{\varepsilon\to 0} r_\varepsilon = 0$ for smooth $\mathbf{u},\omega$, one can obtain the result for all $\omega \in L^p(\mathbb{T}^2), \mathbf{u} \in W^{1,p}(\mathbb{T}^2)$ by an approximation argument using the estimate in Proposition 1. □

If $\mathbf{u}$ is related to ω by the Biot-Savart formula, $\mathbf{u} = \mathbf{K} * \omega$, then $\tau_\varepsilon(\mathbf{u},\omega) = \boldsymbol{\sigma}_\varepsilon$ in the earlier notation. In particular, we see that

$$\partial_t\omega_\varepsilon + (\mathbf{u}_\varepsilon\boldsymbol{\cdot}\boldsymbol{\nabla})\omega_\varepsilon = r_\varepsilon \tag{3.18}$$

for a weak Euler solution.

Proof of Theorem 2: Using (3.18) we get

$$\partial_t h(\omega_\varepsilon) + (\mathbf{u}_\varepsilon\boldsymbol{\cdot}\boldsymbol{\nabla})h(\omega_\varepsilon) = h'(\omega_\varepsilon)r_\varepsilon. \tag{3.19}$$

It was proved in Theorem 1 that

$$\partial_t h(\omega_\varepsilon) + (\mathbf{u}_\varepsilon\boldsymbol{\cdot}\boldsymbol{\nabla})h(\omega_\varepsilon) \longrightarrow \partial_t h(\omega) + (\mathbf{u}\boldsymbol{\cdot}\boldsymbol{\nabla})h(\omega) \tag{3.20}$$

in the sense of distributions for all such h. Furthermore, for any $h \in C^1$ with $h' \in L^\infty$,

$$\|h'(\omega_\varepsilon)r_\varepsilon\|_{L^1} \to 0. \tag{3.21}$$

Having proved that (2.9) holds for h with $h' \in L^\infty$ we then extend it to the general h in the theorem statement by an approximation argument, as in Corollaries II.1-2 in [15]. $\square$

Remark: The smoothness assumed in the earlier proof of [7] is not necessary to obtain conservation, but only to provide an estimate of the rate of the vanishing of the flux. With the assumption that $\omega \in L^p(0,T;B_p^{s,\infty}(\mathbb{T}^2))$ the bounds above can be improved as follows. General estimates in Besov spaces give $\|\boldsymbol{\nabla}\omega_\varepsilon(t)\|_p \leq C\varepsilon^{s-1}\|\omega(t)\|_{B_p^{s,\infty}}$ and $\sup_{|\ell|<\varepsilon}\|\Delta_\ell\omega(t)\|_p \leq \varepsilon^s\|\omega(t)\|_{B_p^{s,\infty}}$. See [12], or Appendix C of [7]. Just as in [7] this gives

$$\|Z_{h,\varepsilon}(\omega(t))\|_1 \leq (\text{const.})\varepsilon^{2s}\|\omega(t)\|_p^{r-1}\|\omega(t)\|^2_{B_p^{s,\infty}} \leq (\text{const.})\varepsilon^{2s}\|\omega(t)\|^{r+1}_{B_p^{s,\infty}} \tag{3.22}$$

Because $r+1<p$, integrating over $t\in[0,T]$ gives

$$\|Z_{h,\varepsilon}(\omega)\|_{L^1([0,T]\times\mathbb{T}^2)} \leq (\text{const.})\varepsilon^{2s}\|\omega\|^{r+1}_{L^p(0,T;B_p^{s,\infty}(\mathbb{T}^2))}. \tag{3.23}$$

Thus, $\lim_{\varepsilon\to 0} Z_{h,\varepsilon}(\omega)=0$ as before, but with an estimate of the rate. The bound $O(\varepsilon^{2s})$ is in agreement with the estimate given by the heuristic argument in the Introduction.

2.3. Proof of Theorem 3

Theorem 3 is a consequence of the following technical lemma:

Proposition 1 *Consider a sequence $\{\omega^\varepsilon|\varepsilon>0\}$ and $\mathbf{u}^\varepsilon=\mathbf{K}*\omega^\varepsilon$ given by the Biot-Savart formula, with the following properties:*

$$\sup_{\varepsilon>0}\|\omega^\varepsilon\|_{L^r(0,T;B_p^{0,\infty}(\mathbb{T}^2))}<\infty \tag{3.24}$$

for $r,p\in[2,\infty]$, and

$$\sup_{\varepsilon>0}\|\mathbf{u}^\varepsilon\|_{\mathrm{Lip}(0,T;H^{-L}(\mathbb{T}^2))}<\infty \tag{3.25}$$

for some $L>3$. Then, there exist ω and $\mathbf{u}=\mathbf{K}*\omega$ with

$$\omega\in L^r(0,T;B_p^{0,\infty}(\mathbb{T}^2)) \tag{3.26}$$

and

$$\mathbf{u}\in \mathrm{Lip}(0,T;H^{-L}(\mathbb{T}^2)). \tag{3.27}$$

and, furthermore, there exists a subsequence of $\omega^\varepsilon, \mathbf{u}^\varepsilon$ *along which*

$$\omega^\varepsilon \rightharpoonup \omega \ \ \textit{weak} - * \ \text{ in } \ L^r(0,T;B_p^{0,\infty}(\mathbb{T}^2)), \tag{3.28}$$

$$\omega^\varepsilon \to \omega \ \ \textit{strong in} \ \ L^r(0,T;W^{-s,q}(\mathbb{T}^2)), \tag{3.29}$$

for some $q > p$ *and* $s > 2\left(\frac{1}{p} - \frac{1}{q}\right)$ *and for* $t = \min\{r, q\} \geq 2$,

$$\mathbf{u}^\varepsilon \to \mathbf{u} \ \ \textit{strong in} \ \ L^t([0,T] \times \mathbb{T}^2). \tag{3.30}$$

Proof: The first statement (3.28) on weak-* convergence of ω^ε to $\omega \in L^r(0,T;B_p^{0,\infty}(\mathbb{T}^2))$ is a simple consequence of the Banach-Alaoglu theorem.

We derive the second statement from the Aubin-Lions compactness criterion (see [30], Theorem 5.1 or [31], Theorem III.2.1). Note first that there is the continuous embedding $B_p^{0,\infty}(\mathbb{T}^2) \subset B_q^{-s',q}(\mathbb{T}^2) = W^{-s',q}(\mathbb{T}^2)$ for each $q > p$ and $s' > 2\left(\frac{1}{p} - \frac{1}{q}\right)$ (see [12], Theorem 2.7.1 and Prop.2.3.2/2). Therefore, from (3.24),

$$\sup_{\varepsilon > 0} \|\omega^\varepsilon\|_{L^r(0,T;W^{-s',q}(\mathbb{T}^2))} < \infty \tag{3.31}$$

On the other hand, from (3.25),

$$\sup_{\varepsilon > 0} \left\| \frac{d\omega^\varepsilon}{dt} \right\|_{L^\infty(0,T;H^{-(L+1)}(\mathbb{T}^2))}. \tag{3.32}$$

Furthermore, for $s > s'$ and $L + 1 > s$ there are continuous embeddings

$$W^{-s',q}(\mathbb{T}^2)) \subset W^{-s,q}(\mathbb{T}^2)) \subset H^{-(L+1)}(\mathbb{T}^2), \tag{3.33}$$

and the first embedding is compact by the Rellich-Kondrachov theorem (see [32], Chapter 12). Hence, we conclude that $\{\omega^\varepsilon | \varepsilon > 0\}$ is compact in $L^r(0,T;W^{-s,q}(\mathbb{T}^2))$ and contains a strongly convergent subsequence.

To obtain the third result we remark that one may choose $0 < s < 1$ and that the mapping $\omega \mapsto \mathbf{u} = \mathbf{K} * \omega$ is continuous from $W^{-s,q}(\mathbb{T}^2)$ into $W^{1-s,q}(\mathbb{T}^2)$, because of the continuity of the singular integral operator $\mathbf{T}(\omega) = (\boldsymbol{\nabla}\mathbf{K}) * \omega$ from $W^{-s,q}(\mathbb{T}^2)$ into itself (for example, see [33],

Theorem 3.2.1) and the bound $\|\mathbf{u}\|_{W^{1-s,q}(\mathbb{T}^2)} \leq (\text{const.}) \left[\|\mathbf{u}\|_{W^{-s,q}(\mathbb{T}^2)} + \|\boldsymbol{\nabla}\mathbf{u}\|_{W^{-s,q}(\mathbb{T}^2)}\right]$ (see [12], Theorem 2.3.8). Of course, convergence of $\mathbf{u}^\varepsilon \to \mathbf{u}$ strong in $L^r(0,T;W^{1-s,q}(\mathbb{T}^2))$ implies at once convergence strong in $L^t([0,T]\times\mathbb{T}^2)$. $\square$

Proof of Theorem 3: The proof is very straightforward and quite similar to that of DiPerna and Majda in [11] for $\omega_0 \in L^p$ with $p \geq 2$ (the easier case than $1 < p < 2$). In fact, the Lipschitz estimate in time

$$\sup_{\nu>0} \|\mathbf{u}^\nu\|_{\mathrm{Lip}(0,T;H^{-L}(\mathbb{T}^2))} < \infty \tag{3.34}$$

holds for the $2D$ Leray solutions with initial energy finite, $E_0 := \frac{1}{2}\|\mathbf{u}_0\|_{L^2}^2 < \infty$, which is part of our assumption. See Section 2A and Appendix A of [11], for example. But, in that case, the Proposition 1 applies, with $r = p = 2$. The limiting velocity $\mathbf{u}$ is easily seen to be a weak solution of the 2D Euler equation in the velocity-pressure formulation, because of the third result (3.30), the strong L^t convergence $\mathbf{u}^\nu \to \mathbf{u}$ with $t > 2$. Obviously, more general versions of Theorem 3 for any $r, p \in [2,\infty]$ could be proved, with (3.24) replacing (2.19) in the hypothesis.

The statements on the weak solutions in the vorticity-velocity formulation follow from arguments very similar to those earlier in Lemma 1, but now using the density of $C^\infty(\mathbb{T}^2)$ in $B_q^{0,1}(\mathbb{T}^2)$ ([12], Theorem 2.3.3). So, we just verify the required regularity of $\mathbf{u}$. Because of the hypothesis on ω, $\mathbf{u} \in L^r(0,T;B_p^{1,\infty}(\mathbb{T}^2))$. Indeed, using the Calderón-Zygmund inequality it is easy to show that $\boldsymbol{\nabla}\mathbf{u} \in L^r(0,T;B_p^{0,\infty}(\mathbb{T}^2))$, and this is equivalent to the first statement ([12], Theorem 2.3.8). Then, for any $p'' < p'$ with p' defined by $\frac{1}{p'} = \frac{1}{p} - \frac{1}{2}$, one has the continuous embedding $B_p^{1,\infty}(\mathbb{T}^2) \subset B_{p''}^{0,1}(\mathbb{T}^2)$. In fact, $B_p^{1,\infty}(\mathbb{T}^2) \subset B_{p''}^{\varepsilon,\infty}(\mathbb{T}^2)$ for $\frac{1}{p''} := \frac{1}{p'} + \frac{\varepsilon}{2}$ for any small $\varepsilon > 0$ ([12], Theorem 2.7.1), but then $B_{p''}^{\varepsilon,\infty}(\mathbb{T}^2) \subset B_{p''}^{0,1}(\mathbb{T}^2)$ by an elementary imbedding ([12], Prop. 2.3.2/2). Now, precisely for $p > \frac{4}{3}$, one has $\frac{1}{p'} < \frac{1}{q}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Thus, it is possible to choose $p'' > q$ but still $p'' < p'$. Also, with t defined by $\frac{1}{r} + \frac{1}{t} = 1$, $r \geq t$ for $r \geq 2$. In that case, $\mathbf{u} \in L^t(0,T;B_q^{0,1}(\mathbb{T}^2))$, whereas $L^r(0,T;B_p^{0,\infty}(\mathbb{T}^2)) = [L^t(0,T;B_q^{0,1}(\mathbb{T}^2))]^*$, the Banach dual ([12], Theorem 2.11.2). To conclude the proof, we just note that, if $\psi \in C_0^\infty([0,T]\times\mathbb{T}^2)$, then also $(\partial_t + \mathbf{u}\boldsymbol{\cdot}\boldsymbol{\nabla})\psi \in L^t(0,T;B_q^{0,1}(\mathbb{T}^2))$ (see [12], Lemma 3.3.1). $\square$

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